The Inherent Robustness of a New Approach to Adaptive Control

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Abstract— **Recently it has been shown how to carry out adaptive control for an LTI plant so that the effect of the initial condition decays exponentially to zero and so that the input-output behavior enjoys a convolution bound. This, in turn, has been leveraged to prove, in several special cases, that the closed-loop system is robust in the sense that both of these properties are maintained in the presence of a small amount of parameter time-variation and unmodelled dynamics. The goal of this paper is to show that this robustness property is true for a general adaptive controller which may include multi-estimators; the immediate ramification is that if we are able to prove exponential stability and a convolution bound for the case of fixed plant parameters, then robustness comes for free.**

I. INTRODUCTION

In control system design, a common requirement is that the closed-loop system not only be stable, but also be robust, in the sense that the desired closed-loop properties are maintained, at the very least, in the presence of small time-variations in the plant parameters and a small amount of unmodelled dynamics. Of course, if the plant and controller are both linear and time-invariant, and the desired objective is closed-loop stability, then such robustness follows from the Small Gain Theorem [27] and the study of time-varying linear systems [1]. On the other hand, if either the plant or controller is nonlinear, this is often not the case and/or it is not easy to prove.

One special class of nonlinear controllers is that of adaptive controllers, wherein the controller learns about the plant as time progresses. While adaptive control has been studied as far back as the 1950s, the first general proofs that parameter adaptive controllers work came around 1980, e.g. see [2], [15], [3], [18], and [17]. However, the original controllers are typically not robust to unmodelled dynamics, do not tolerate time-variations well, and do not handle noise (or disturbances) well, e.g. see [19]. A number of small controller design changes were proposed, such as the use of signal normalization, deadzones, and $\sigma-$modification, and projection onto a convex set of admissible parameters e.g. see [8], [9], [22], [6], [4], [26], [25], [16], [24], [23] and [7]. However, in general these redesigned controllers provide asymptotic stability but not exponential stability, with no bounded gain on the noise, let alone a convolution bound.

This brings us to our recent work on using the original, unmodified, projection algorithm to prove enhanced behavior. It has been proven, in a variety of settings, that if discrete-time adaptive control is carried out using this algorithm then exponential stability and a convolution bound on the closed-loop behavior can be proven—see [10], [11], [12], [13], [14], [20] and [21]; hence, the closed-loop system acts 'linear-like'. We have shown, in the first-order one step-ahead case [10] and the pole placement case with a single estimator [12], that this approach is robust; this is **proven in a modular fashion**—we leverage the exponential stability and the convolution bounds proven for the nominal plant model without reopening its proof. This differs markedly from the approach that most papers on robust adaptive control adopt: there one proves robustness by taking the proof for the ideal case and creating a more complicated version with a time-varying plant with some unmodelled dynamics added. The goal of this paper is to prove that this modularity property holds in a very general adaptive control setting; modularity is a highly desirable property, since it allows you to focus on analyzing the ideal plant model, knowing that robustness will come for free.

To this end, here we consider a class of finite-dimensional, nonlinear plant and adaptive controller combinations; if exponential stability and a convolution bound holds, then we prove that tolerance to small time-variations in the plant parameters and a small amount of unmodelled dynamics follows. An immediate application of this result is to prove robustness of our recently designed high order one-step-ahead controller [13] and our multi-estimator switching adaptive controllers presented in [12], [20] and [21]. This result should also prove useful in extending our work on the adaptive control of LTI plants [10] [11], [12], [20], [13], [21] to that of nonlinear plants, allowing us to focus on the ideal plant model in the analysis.

We denote $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{N}$ as the sets of integers, non-negative integers and natural numbers, respectively. We will denote the Euclidean-norm of a vector and the induced norm of a matrix by the subscript-less default notation $\|\cdot\|$. We let $\mathbb{S}(\mathbb{R}^{p\times q})$ denote the set of $\mathbb{R}^{p\times q}$-valued sequences. We also let $\boldsymbol{\ell}_\infty(\mathbb{R}^{p\times q})$ denote the set of $\mathbb{R}^{p\times q}$-valued *bounded* sequences. If $\Omega \subset \mathbb{R}^{p\times q}$ is a bounded set, we define $\|\Omega\| := \sup_{x\in\Omega}\|x\|$.

Throughout this paper, we say that a function $\Gamma : \mathbb{R}^p \to \mathbb{R}^q$ has a *bounded gain* if there exists a $\nu > 0$ such that for all $x \in \mathbb{R}^p$, we have $\|\Gamma(x)\| \le \nu\|x\|$; the smallest such ν is the gain, and is denoted by $\|\Gamma\|$.

For a closed and convex set $\Omega \subset \mathbb{R}^p$, the function $\text{Proj}_\Omega\{\cdot\} : \mathbb{R}^p \to \Omega$ denotes the projection onto Ω; it is well known that the function Proj_Ω is well defined.

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II. The Setup

Here the nominal plant is multi-input multi-output[1] with finite memory and an additive disturbance, such that the uncertain plant parameter enters linearly. To this end, with an output $y(t) \in \mathbb{R}^r$, an input $u(t) \in \mathbb{R}^m$, a disturbance $w(t) \in \mathbb{R}^r$, a modeling parameter of

$$\theta^* \in \mathcal{S} \subset \mathbb{R}^{p \times r},$$

and a vector of input-output data of the form

$$\phi(t) = \begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-n_y+1) \\ u(t) \\ u(t-1) \\ \vdots \\ u(t-n_u+1) \end{bmatrix} \in \mathbb{R}^{n_y \cdot r + n_u \cdot m},$$

we consider the plant

$$y(t+1) \;=\; {\theta^*}^\top f\big(\phi(t)\big) + w(t), \quad \phi(t_0) = \phi_0; \quad (1)$$

we assume that $f : \mathbb{R}^{n_y \cdot r + n_u \cdot m} \to \mathbb{R}^p$ has a bounded gain and that $\mathcal{S}$ is a bounded set; both requirements are reasonable given that we will require uniform bounds in our analysis. We represent this system by the pair $\big(f, \mathcal{S}\big)$.

Here we consider a large class of controllers which subsumes LTI ones as well as a large class of adaptive ones. To this end, we consider a controller with its state partitioned into two parts:

- $z(t) \in \mathbb{R}^{l_1}$ and
- $\hat{\theta}(t) \in \mathbb{R}^{l_2}$,

with an exogenous signal $r(t) \in \mathbb{R}^r$ (typically a reference signal), and with equations of the form[2]

$$z(t+1) = g_1\big(z(t), \hat{\theta}(t), \phi(t), r(t), t, t_0\big), \;\; z(t_0) = z_0 \quad (2a)$$

$$\hat{\theta}(t+1) = g_2\big(z(t), \hat{\theta}(t), \phi(t), r(t), t, t_0\big), \;\; \hat{\theta}(t_0) = \theta_0 \quad (2b)$$

$$u(t) = h\big(z(t), \hat{\theta}(t), \phi(t-1), y(t), r(t), t, t_0\big). \quad (2c)$$

With $\Omega \subset \mathbb{R}^{l_2}$ a bounded set, we assume that

$$g_2 : \mathbb{R}^{l_1} \times \Omega \times \mathbb{R}^{n_y \cdot r + n_u \cdot m} \times \mathbb{R}^r \times \mathbb{Z} \times \mathbb{Z} \longrightarrow \Omega,$$

i.e. if $\hat{\theta}$ is initialized in Ω, then it remains in Ω throughout.

Remark 1. *This class subsumes finite-dimensional LTI controllers: simply set $l_2 = 0$ so that the sub-state $\hat{\theta}(t)$ disappears, and make the functions g_1 and h be linear.*

Remark 2. *This class subsumes many adaptive controllers: simply set $l_1 = 0$ and let $\hat{\theta}(t)$ be the state of a parameter estimator constrained to the set Ω. If we are using multiple estimators, then the dimension of $\hat{\theta}(t)$ is typically larger than that of θ^*.*

[1]This model is more general than we need in our examples, but the cost of this is minimal.

[2]The inclusion of t_0 in these equations allows us to include performance-based switching adaptive controllers.

We now provide a definition of the desired linear-like closed-loop property:

Definition 1. *We say that* (2) **provides exponential stability and a convolution bound for** $\big(f, \mathcal{S}\big)$ *with gain $c \geq 1$ and decay rate $\lambda \in (0,1)$ if, for every $\theta^* \in \mathcal{S}$, $t_0 \in \mathbb{Z}$, $\phi_0 \in \mathbb{R}^{n_y \cdot r + n_u \cdot m}$, $z_0 \in \mathbb{R}^{l_1}$, $\theta_0 \in \Omega \subset \mathbb{R}^{l_2}$, $w \in \mathbb{S}(\mathbb{R}^r)$ and $r \in \mathbb{S}(\mathbb{R}^r)$, when* (2) *is applied to* (1)*, the following holds:*

$$\left\| \begin{bmatrix} \phi(t) \\ z(t) \end{bmatrix} \right\| \leq c\lambda^{t-\tau} \left\| \begin{bmatrix} \phi(\tau) \\ z(\tau) \end{bmatrix} \right\| + \sum_{j=\tau}^{t-1} c\lambda^{t-j-1}(\|r(j)\| + \|w(j)\|) + c\|r(t)\|, \quad t \geq \tau \geq t_0. \quad (3)$$

Remark 3. *Most adaptive controllers do not provide exponential stability and a convolution bound. As far as the authors are aware, the only ones which do are those in our recent work [10], [11], [12], [20], [13], [21], [14].*

III. Tolerance to Time-Variation

We now consider plants with a possibly time-varying parameter vector $\theta^*(t)$ instead of a static θ^*:

$$y(t+1) = \theta^*(t)^\top f\big(\phi(t)\big) + w(t), \quad \phi(t_0) = \phi_0. \quad (4)$$

With $c_0 \geq 0$ and $\epsilon > 0$, let $s(\mathcal{S}, c_0, \epsilon)$ denote the subset of $\ell_\infty(\mathbb{R}^{p \times r})$ whose elements θ^* satisfy:

- $\theta^*(t) \in \mathcal{S}$ for every $t \in \mathbb{Z}$,
- and

$$\sum_{t=t_1}^{t_2-1} \|\theta^*(t+1) - \theta^*(t)\| \leq c_0 + \epsilon(t_2 - t_1), \quad t_2 > t_1, \; t_1 \in \mathbb{Z}.$$

The above time-variation model encompasses both slow variations and/or occasional jumps; this class is well-known in the adaptive control literature, e.g. see [5]. We can extend Definition 1 in a natural way to handle time-variations.

Definition 2. *We say that* (2) **provides exponential stability and a convolution bound for** $\big(f, s(\mathcal{S}, c_0, \epsilon)\big)$ *with gain $c \geq 1$ and decay rate $\lambda \in (0,1)$ if, for every $\theta^* \in s(\mathcal{S}, c_0, \epsilon)$, $t_0 \in \mathbb{Z}$, $\phi_0 \in \mathbb{R}^{n_y \cdot r + n_u \cdot m}$, $z_0 \in \mathbb{R}^{l_1}$, $\theta_0 \in \Omega \subset \mathbb{R}^{l_2}$, $w \in \mathbb{S}(\mathbb{R}^r)$ and $r \in \mathbb{S}(\mathbb{R}^r)$, when* (2) *is applied to* (4)*, the following holds:*

$$\left\| \begin{bmatrix} \phi(t) \\ z(t) \end{bmatrix} \right\| \leq c\lambda^{t-\tau} \left\| \begin{bmatrix} \phi(\tau) \\ z(\tau) \end{bmatrix} \right\| + \sum_{j=\tau}^{t-1} c\lambda^{t-j-1}(\|r(j)\| + \|w(j)\|) + c\|r(t)\|, \quad t \geq \tau \geq t_0. \quad (5)$$

We now will show that if a controller (2) provides exponential stability and a convolution bound for the plant (1), then the same will be true for the time-varying plant (4), as long as ϵ is small enough.

Theorem 1. *Suppose that the controller (2) provides exponential stability and a convolution bound for $(f, \mathcal{S})$ with gain $c \geq 1$ and decay rate $\lambda \in (0,1)$. Then for every $\lambda_1 \in (\lambda, 1)$ and $c_0 > 0$, there exist a $c_1 \geq c$ and $\epsilon > 0$ so that (2) provides exponential stability and a convolution bound for $\big(f, s(\mathcal{S}, c_0, \epsilon)\big)$ with gain c_1 and decay rate λ_1.*

Remark 4. *This proof is based, in part, on the proof of Theorem 2 of [12], which deals with a much simpler setup.*

Proof of Theorem 1. Suppose the controller (2) provides exponential stability and a convolution bound for (1) with gain $c \geq 1$ and a decay rate of λ. Fix $\lambda_1 \in (\lambda, 1)$ and $c_0 > 0$; let $t_0 \in \mathbb{Z}$, $\phi_0 \in \mathbb{R}^{n_y \cdot r + n_u \cdot m}$, $z_0 \in \mathbb{R}^{l_1}$, $\theta_0 \in \Omega$, $w \in \mathbb{S}(\mathbb{R}^r)$ and $r \in \mathbb{S}(\mathbb{R}^r)$ be arbitrary.

Now fix $m \in \mathbb{N}$ to be any number satisfying

$$m \geq \frac{\ln(c) + \frac{4c_0 c\|f\|}{\lambda_1 - \lambda}\left[\ln\left(1 + 2c\|f\|\|\mathcal{S}\|\right) + \ln(2) - \ln(\lambda + \lambda_1)\right]}{\ln(2\lambda_1) - \ln(\lambda + \lambda_1)},$$

and then set $\epsilon = \frac{c_0}{m^2}$; let $\theta^* \in s(\mathcal{S}, c_0, \epsilon)$ be arbitrary and apply the controller (2) to the time-varying plant (4). To proceed, we analyze the closed-loop system behavior on intervals of length m, which we further analyze in groups of m^2.

To proceed, let $\bar{t} \geq t_0$ be arbitrary. Define a sequence $\{\bar{t}_i\}$ by $\bar{t}_i = \bar{t} + im$ for $i \in \mathbb{Z}^+$. We can rewrite the time-varying plant as

$$y(t+1) = \theta^*(\bar{t}_i)^\top f(\phi(t)) + w(t) + \underbrace{\left[\theta^*(t) - \theta^*(\bar{t}_i)\right]^\top f(\phi(t))}_{=:\tilde{n}_i(t)}, \quad t \in [\bar{t}_i, \bar{t}_{i+1}).$$

On the interval $[\bar{t}_i, \bar{t}_{i+1}]$, we can regard the plant as time-invariant, but with an extra disturbance; so by hypothesis,

$$\left\| \begin{bmatrix} \phi(t) \\ z(t) \end{bmatrix} \right\| \leq c\lambda^{t-\bar{t}_i} \left\| \begin{bmatrix} \phi(\bar{t}_i) \\ z(\bar{t}_i) \end{bmatrix} \right\| + \sum_{j=\bar{t}_i}^{t-1} c\lambda^{t-j-1}(\|r(j)\| + \|w(j)\| + \|\tilde{n}_i(j)\|) + c\|r(t)\|,$$
$$t \in [\bar{t}_i, \bar{t}_{i+1}], \; i \in \mathbb{Z}^+. \tag{6}$$

To analyze this difference inequality, we first construct an associated difference equation:

$$\psi(t+1) = \lambda\psi(t) + \|r(t)\| + \|w(t)\| + \|\tilde{n}_i(t)\|, \quad t \in [\bar{t}_i, \bar{t}_{i+1}),$$

with an initial condition of

$$\psi(\bar{t}_i) = \left\| \begin{bmatrix} \phi(\bar{t}_i) \\ z(\bar{t}_i) \end{bmatrix} \right\|.$$

Using the fact that $c \geq 1$, it is straightforward to prove that

$$\left\| \begin{bmatrix} \phi(t) \\ z(t) \end{bmatrix} \right\| \leq c\psi(t) + c\|r(t)\|, \;\; t \in [\bar{t}_i, \bar{t}_{i+1}]. \tag{7}$$

Now we analyze this equation for $i = 0, 1, \dots, m-1$.

Case 1: $\|\tilde{n}_i(t)\| \leq \frac{\lambda_1 - \lambda}{2c}\|\phi(t)\|$ for all $t \in [\bar{t}_i, \bar{t}_{i+1})$.

Using the above bound (7) and the fact that $\lambda_1 - \lambda \in (0,1)$, we obtain

$$\begin{aligned}
\psi(t+1) &\leq \lambda\psi(t) + \|r(t)\| + \|w(t)\| + \|\tilde{n}_i(t)\| \\
&\leq \lambda\psi(t) + \|r(t)\| + \|w(t)\| + \tfrac{\lambda_1-\lambda}{2c}\|\phi(t)\| \\
&\leq \lambda\psi(t) + \|r(t)\| + \|w(t)\| + \\
&\qquad \tfrac{\lambda_1-\lambda}{2}\left(\psi(t) + \|r(t)\|\right) \\
&\leq \tfrac{\lambda_1+\lambda}{2}\psi(t) + 2\|r(t)\| + \|w(t)\|, \quad t \in [\bar{t}_i, \bar{t}_{i+1}),
\end{aligned}$$

which means that

$$|\psi(t)| \leq \left(\tfrac{\lambda_1+\lambda}{2}\right)^{t-\bar{t}_i}|\psi(\bar{t}_i)| + \sum_{j=\bar{t}_i}^{t-1}\left(\tfrac{\lambda_1+\lambda}{2}\right)^{t-j-1}\left(2\|r(j)\| + \|w(j)\|\right),$$
$$t = \bar{t}_i, \bar{t}_i + 1, \dots, \bar{t}_{i+1}. \tag{8}$$

This, in turn, implies that there exists $c_2 \geq 2c$ so that

$$\left\| \begin{bmatrix} \phi(\bar{t}_{i+1}) \\ z(\bar{t}_{i+1}) \end{bmatrix} \right\| \leq c\left(\tfrac{\lambda_1+\lambda}{2}\right)^m \left\| \begin{bmatrix} \phi(\bar{t}_i) \\ z(\bar{t}_i) \end{bmatrix} \right\| + \sum_{j=\bar{t}_i}^{\bar{t}_{i+1}-1} c_2\left(\tfrac{\lambda_1+\lambda}{2}\right)^{\bar{t}_{i+1}-j-1}\left(\|r(j)\| + \|w(j)\|\right) + c_2\|r(\bar{t}_{i+1})\|. \tag{9}$$

Case 2: $\|\tilde{n}_i(t)\| > \frac{\lambda_1-\lambda}{2c}\|\phi(t)\|$ for some $t \in [\bar{t}_i, \bar{t}_{i+1})$.

Since $\theta^*(t) \in \mathcal{S}$ for $t \geq t_0$, we see that

$$\|\tilde{n}_i(t)\| \leq 2\|f\|\|\mathcal{S}\| \times \|\phi(t)\|, \qquad t \in [\bar{t}_i, \bar{t}_{i+1}).$$

This means that

$$\begin{aligned}
\psi(t+1) &\leq \lambda\psi(t) + \|r(t)\| + \|w(t)\| + \|\tilde{n}_i(t)\| \\
&\leq \lambda\psi(t) + \|r(t)\| + \|w(t)\| + 2\|f\|\|\mathcal{S}\|\|\phi(t)\| \\
&\leq \underbrace{(1 + 2c\|f\|\|\mathcal{S}\|)}_{=:\gamma_3}\psi(t) + \\
&\qquad (1 + 2c\|f\|\|\mathcal{S}\|)\|r(t)\| + \|w(t)\|, \quad t \in [\bar{t}_i, \bar{t}_{i+1}),
\end{aligned}$$

which means that

$$|\psi(t)| \leq \gamma_3^{t-\bar{t}_i}|\psi(\bar{t}_i)| + \sum_{j=\bar{t}_i}^{t-1}\gamma_3^{t-j-1}\left(\gamma_3\|r(j)\| + \|w(j)\|\right),$$
$$t = \bar{t}_i, \bar{t}_i + 1, \dots, \bar{t}_{i+1}. \tag{10}$$

Setting $t = \bar{t}_{i+1}$ and using (7) yields

$$\begin{aligned}
\left\| \begin{bmatrix} \phi(\bar{t}_{i+1}) \\ z(\bar{t}_{i+1}) \end{bmatrix} \right\| &\leq c\gamma_3^m \left\| \begin{bmatrix} \phi(\bar{t}_i) \\ z(\bar{t}_i) \end{bmatrix} \right\| + \sum_{j=\bar{t}_i}^{\bar{t}_{i+1}-1} c\gamma_3^{\bar{t}_{i+1}-j-1}\left(\gamma_3\|r(j)\| + \|w(j)\|\right) + c\|r(\bar{t}_{i+1})\| \\
&\leq c\gamma_3^m \left\| \begin{bmatrix} \phi(\bar{t}_i) \\ z(\bar{t}_i) \end{bmatrix} \right\| + c\gamma_3\left(\tfrac{2\gamma_3}{\lambda_1+\lambda}\right)^m \times \\
&\qquad \sum_{j=\bar{t}_i}^{\bar{t}_{i+1}-1}\left(\tfrac{\lambda_1+\lambda}{2}\right)^{\bar{t}_{i+1}-j-1}\left(\|r(j)\| + \|w(j)\|\right) + c\|r(\bar{t}_{i+1})\|.
\end{aligned} \tag{11}$$

This completes Case 2.

At this point we combine Case 1 and 2. We would like to analyze m intervals of length m. On the interval $[\bar{t},\bar{t}+m^2]$, there are m subintervals of length m; furthermore, because of the choice of ϵ we have that

$$\sum_{j=\bar{t}}^{\bar{t}+m^2-1} \|\theta^*(j+1)-\theta^*(j)\| \le c_0+\epsilon m^2 \le 2c_0.$$

It is easy to see that there are at most $N_1 := \frac{4c_0 c\|f\|}{\lambda_1-\lambda}$ subintervals which fall into the category of Case 2, with the remainder falling into the category of Case 1; it is clear from the formula for m that $m > N_1$. If we use (9) and (11) to analyze the behavior of the closed-loop system on the interval $[\bar{t},\bar{t}+m^2]$, we end up with a crude bound of

$$\left\| \begin{bmatrix} \phi(\bar{t}+m^2) \\ z(\bar{t}+m^2) \end{bmatrix} \right\| \le c^m \gamma_3^{N_1 m} \left(\tfrac{\lambda_1+\lambda}{2}\right)^{m(m-N_1)} \left\| \begin{bmatrix} \phi(\bar{t}) \\ z(\bar{t}) \end{bmatrix} \right\| + 2m \left(\tfrac{2\gamma_3}{\lambda_1+\lambda}\right)^m (c_2\gamma_3^{m+1})^m \left(\tfrac{2}{\lambda_1+\lambda}\right)^{(m+1)m} \times \sum_{j=\bar{t}}^{\bar{t}+m^2-1} \left(\tfrac{\lambda_1+\lambda}{2}\right)^{\bar{t}+m^2-j-1} (\|r(j)\|+\|w(j)\|) + c_2\|r(\bar{t}+m^2)\|. \quad (12)$$

From the choice of m above, it is easy to show that

$$m^2 \ln\left(\tfrac{2\lambda_1}{\lambda_1+\lambda}\right) \ge m\ln(c)+N_1 m\ln(\gamma_3)+N_1 m \ln\left(\tfrac{2}{\lambda+\lambda_1}\right);$$

this immediately implies that

$$c^m \gamma_1^{N_1 m} \left(\frac{\lambda_1+\lambda}{2}\right)^{m(m-N_1)} \le \lambda_1^{m^2}.$$

Since $\frac{\lambda_1+\lambda}{2} < \lambda_1$, it follows from (12) that there exists a constant γ_4 so that

$$\left\| \begin{bmatrix} \phi(\bar{t}+m^2) \\ z(\bar{t}+m^2) \end{bmatrix} \right\| \le \lambda_1^{m^2} \left\| \begin{bmatrix} \phi(\bar{t}) \\ z(\bar{t}) \end{bmatrix} \right\| + \gamma_4 \sum_{j=\bar{t}}^{\bar{t}+m^2-1} \lambda_1^{\bar{t}+m^2-j-1} (\|r(j)\|+\|w(j)\|) + \gamma_4\|r(\bar{t}+m^2)\|. \quad (13)$$

Now let $\tau \ge t_0$ be arbitrary. By setting $\bar{t} = \tau, \tau+m^2, \tau+2m^2, \ldots$, in succession, it follows from (13) that

$$\left\| \begin{bmatrix} \phi(\tau+qm^2) \\ z(\tau+qm^2) \end{bmatrix} \right\| \le \lambda_1^{qm^2} \left\| \begin{bmatrix} \phi(\tau) \\ z(\tau) \end{bmatrix} \right\| + \gamma_4 \sum_{j=\tau}^{\tau+qm^2-1} \lambda_1^{\tau+qm^2-j-1} (\|r(j)\|+\|w(j)\|) + \gamma_4\|r(\tau+qm^2)\|, \qquad q\in\mathbb{Z}^+. \quad (14)$$

So $\begin{bmatrix} \phi(t) \\ z(t) \end{bmatrix}$ is well-behaved at $t=\tau,\tau+m^2,\tau+2m^2$, etc; we can use (8) of Case 1, (10) of Case 2 and (7) to prove that nothing untoward happens between these times. We conclude that there exists a constant γ_5 so that

$$\left\| \begin{bmatrix} \phi(t) \\ z(t) \end{bmatrix} \right\| \le \gamma_5 \lambda_1^{t-\tau} \left\| \begin{bmatrix} \phi(\tau) \\ z(\tau) \end{bmatrix} \right\| + \gamma_5 \sum_{j=\tau}^{t-1} \lambda_1^{t-j-1} (\|r(j)\|+\|w(j)\|) + \gamma_5\|r(t)\|, \quad t\ge\tau.$$

Since $\tau \ge t_0$ is arbitrary, the desired bound is proven. ∎

IV. Tolerance to Unmodelled Dynamics

We now consider the time-varying plant (4) with the term $d_\Delta(t)\in\mathbb{R}^r$ added to represent unmodelled dynamics:

$$y(t+1) = \theta^*(t)^\top f\big(\phi(t)\big) + w(t) + d_\Delta(t), \quad \phi(t_0)=\phi_0. \quad (15)$$

Here we consider (a generalized version of) a class of unmodelled dynamics which is common in the adaptive control literature—see [6] and [12]. With $g:\mathbb{R}^{n_y\cdot r+n_u\cdot m}\to\mathbb{R}$ a map with a bounded gain, $\beta\in(0,1)$ and $\mu>0$, we consider

$$\mathfrak{w}(t+1) = \beta\mathfrak{w}(t)+\beta|g(\phi(t))|, \quad \mathfrak{w}(t_0)=\mathfrak{w}_0 \quad (16a)$$

$$\|d_\Delta(t)\| \le \mu\mathfrak{w}(t)+\mu|g(\phi(t))|, \quad t\ge t_0. \quad (16b)$$

It turns out that this model subsumes classical additive uncertainty, multiplicative uncertainty, and uncertainty in a coprime factorization, with side constraints on the pole locations (less than β in magnitude) as well as strict causality; see [12] for a more detailed explanation. We will now show that if the controller (2) provides exponential stability and a convolution bound for $\big(f,s(\mathcal{S},c_0,\epsilon)\big)$, then a degree of tolerance to unmodelled dynamics can be proven.

Theorem 2. *Suppose that the controller (2) provides exponential stability and a convolution bound for $\big(f,s(\mathcal{S},c_0,\epsilon)\big)$ with a gain c_1 and decay rate $\lambda_1\in(0,1)$. Then for every $\beta\in(0,1)$ and $\lambda_2\in(\max\{\lambda_1,\beta\},1)$, there exist $\bar{\mu}>0$ and $c_2>0$ so that for every $\theta^*\in s(\mathcal{S},c_0,\epsilon)$, $\mu\in(0,\bar{\mu})$, $t_0\in\mathbb{Z}$, $\phi_0\in\mathbb{R}^{n_y\cdot r+n_u\cdot m}$, $z_0\in\mathbb{R}^{l_1}$, $\theta_0\in\Omega\subset\mathbb{R}^{l_2}$, and $w,r\in\mathbb{S}(\mathbb{R}^r)$, when the controller (2) is applied to the plant (15) with d_Δ satisfying (16), the following holds:*

$$\left\| \begin{bmatrix} \phi(t) \\ z(t) \\ \mathfrak{w}(t) \end{bmatrix} \right\| \le c_2\lambda_2^{t-t_0} \left\| \begin{bmatrix} \phi_0 \\ z_0 \\ \mathfrak{w}_0 \end{bmatrix} \right\| + \sum_{j=t_0}^{t-1} c_2\lambda_2^{t-j-1}(\|r(j)\|+\|w(j)\|)+c_2\|r(t)\|, \quad t\ge t_0.$$

Remark 5. *If we combine Theorem 1 and 2, we conclude that if the controller (2) provides exponential stability and a convolution bound for the plant $\big(f,\mathcal{S}\big)$, then the same will be true in the presence of a degree of time-variation and unmodelled dynamics, i.e. the approach is* **robust***.*

Proof of Theorem 2. Fix $\beta\in(0,1)$ and $\lambda_2\in(\max\{\lambda_1,\beta\},1)$ and let $\theta^*\in s(\mathcal{S},c_0,\epsilon)$, $t_0\in\mathbb{Z}$, $\phi_0\in\mathbb{R}^{n_y\cdot r+n_u\cdot m}$, $z_0\in\mathbb{R}^{l_1}$, $\theta_0\in\Omega$, $w\in\mathbb{S}(\mathbb{R}^r)$ and $r\in\mathbb{S}(\mathbb{R}^r)$ be arbitrary. So by hypothesis:

$$\left\| \begin{bmatrix} \phi(t) \\ z(t) \end{bmatrix} \right\| \le c_1\lambda_1^{t-\tau} \left\| \begin{bmatrix} \phi(\tau) \\ z(\tau) \end{bmatrix} \right\| +$$

$$\sum_{j=\tau}^{t-1} c_1\lambda_1^{t-j-1}(\|r(j)\|+\|w(j)\|+\|d_\Delta(j)\|)+c_1\|r(t)\|,$$
$$t\geq\tau\geq t_0. \tag{17}$$

To convert this inequality to an equality, we consider the associated difference equations

$$\tilde{\phi}(t+1)=\lambda_1\tilde{\phi}(t)+c_1\|r(t)\|+c_1\|w(t)\|+ c_1\mu\tilde{\mathfrak{w}}(t)+c_1\mu\|g\|\tilde{\phi}(t),\ \tilde{\phi}(t_0)=c_1\left\|\begin{bmatrix}\phi_0\\ z_0\end{bmatrix}\right\|,$$

together with the difference equation based on (16a):

$$\tilde{\mathfrak{w}}(t+1)=\beta\tilde{\mathfrak{w}}(t)+\beta\|g\|\tilde{\phi}(t),\quad \tilde{\mathfrak{w}}(t_0)=|\mathfrak{w}_0|.$$

Using induction together with (17), (16a), and (16b), we can prove that

$$\left\|\begin{bmatrix}\phi(t)\\ z(t)\end{bmatrix}\right\|\leq\tilde{\phi}(t)+c_1\|r(t)\|, \tag{18a}$$
$$|\mathfrak{w}(t)|\leq\tilde{\mathfrak{w}}(t),\qquad t\geq t_0. \tag{18b}$$

If we combine the difference equations for $\tilde{\phi}(t)$ and $\tilde{\mathfrak{w}}(t)$, we obtain

$$\begin{bmatrix}\tilde{\phi}(t+1)\\ \tilde{\mathfrak{w}}(t+1)\end{bmatrix}=\underbrace{\begin{bmatrix}\lambda_1+c_1\|g\|\mu & c_1\mu\\ \beta\|g\| & \beta\end{bmatrix}}_{=:A_{\text{cl}}(\mu)}\begin{bmatrix}\tilde{\phi}(t)\\ \tilde{\mathfrak{w}}(t)\end{bmatrix}+ \begin{bmatrix}c_1\\ 0\end{bmatrix}(\|r(t)\|+\|w(t)\|),\qquad t\geq t_0. \tag{19}$$

Now we see that $A_{\text{cl}}(\mu)\to\begin{bmatrix}\lambda_1 & 0\\ \beta\|g\| & \beta\end{bmatrix}$ as $\mu\to 0$, and this matrix has eigenvalues of $\{\lambda_1,\beta\}$ which are both less that $\lambda_2<1$. Using a standard Lyapunov argument, it is easy to prove that there exist $\bar{\mu}>0$ and $\gamma_1>0$ such that for all $\mu\in(0,\bar{\mu}]$, we have

$$\left\|A_{\text{cl}}(\mu)^k\right\|\leq\gamma_1\lambda_2^k,\qquad k\geq 0;$$

if we use this in (19) and then apply the bound in (18), it follows that

$$\left\|\begin{bmatrix}\phi(t)\\ z(t)\\ \mathfrak{w}(t)\end{bmatrix}\right\|\leq c_1\gamma_1\lambda_2^{t-t_0}\left\|\begin{bmatrix}\phi_0\\ z_0\\ \mathfrak{w}_0\end{bmatrix}\right\|+ \sum_{j=t_0}^{t-1}c_1\gamma_1\lambda_2^{t-j-1}(\|r(j)\|+\|w(j)\|)+c_1\|r(t)\|,\quad t\geq t_0$$

as desired. ■

V. Applications

In this section, we will apply Theorems 1 and 2 to various adaptive control problems which we have solved. In these examples, it turns out that we do not need z as part of the controller.

A. First-Order One-Step-Ahead Adaptive Control

Here we consider the 1st-order linear time-invariant plant

$$y(t+1)=ay(t)+bu(t)+w(t),$$
$$=\underbrace{\begin{bmatrix}a\\ b\end{bmatrix}}_{=:\theta^{*\top}}^{\top}\underbrace{\begin{bmatrix}y(t)\\ u(t)\end{bmatrix}}_{=:\phi(t)}+w(t),\quad y(t_0)=y_0. \tag{20}$$

Here, θ^* is unknown but lies in **closed and bounded set** $\mathcal{S}\subset\mathbb{R}^2$; to ensure controllability we require that $\begin{bmatrix}a\\ 0\end{bmatrix}\notin\mathcal{S}$ for any $a\in\mathbb{R}$. The control objective is to track a reference signal $y^*(t)$ asymptotically; we assume that we know it one step ahead.

In [10] the case of $\mathcal{S}$ being convex is considered and robustness properties are proven. Now we turn to the more general case of $\mathcal{S}$ not convex. This was considered in [20] and exponential stability and a convolution bound was proven, but nothing was proven about robustness. Here we will show that the controller proposed there fits into the framework of this paper, so that Theorems 1 and 2 can be applied. It is proven in [20] that $\mathcal{S}$ can be covered by two convex and compact sets $\mathcal{S}_1$ and $\mathcal{S}_2$ so that, for every $\begin{bmatrix}a\\ b\end{bmatrix}\in\mathcal{S}_1\cup\mathcal{S}_2$ we have that $b\neq 0$. To proceed, we use two parameter estimators—one for $\mathcal{S}_1$ and one for $\mathcal{S}_2$—and then use a switching adaptive controller to switch between the estimates as necessary. For each $i\in\{1,2\}$ and given an estimate $\hat{\theta}_i(t)$ at time $t\geq t_0$, we have a prediction error of

$$e_i(t+1):=y(t+1)-\hat{\theta}_i(t)^\top\phi(t);$$

estimator updates are computed by

$$\check{\theta}_i(t+1)=\begin{cases}\hat{\theta}_i(t)+\dfrac{\phi(t)}{\|\phi(t)\|^2}e_i(t+1) & \text{if } \|\phi(t)\|\neq 0\\ \hat{\theta}_i(t) & \text{otherwise}\end{cases} \tag{21}$$

$$\hat{\theta}_i(t+1)=\text{Proj}_{\mathcal{S}_i}\left\{\check{\theta}_i(t+1)\right\}. \tag{22}$$

We partition $\hat{\theta}_i(t)$ in a natural way by $\hat{\theta}_i(t)=:\begin{bmatrix}\hat{a}_i(t)\\ \hat{b}_i(t)\end{bmatrix}$. We define a switching signal $\sigma:\mathbb{Z}\to\{1,2\}$ to choose which parameter estimates to use in the control law at any point in time. Namely, with $\sigma(t_0)\in\{1,2\}$, the choice is

$$\sigma(t+1)=\arg\min_{i\in\{1,2\}}|e_i(t+1)|,\qquad t\geq t_0, \tag{23}$$

i.e. it is the index corresponding to the smallest prediction error. Next we apply the Certainty Equivalence Principle to yield

$$u(t)=-\frac{\hat{a}_{\sigma(t)}(t)}{\hat{b}_{\sigma(t)}(t)}y(t)+\frac{1}{\hat{b}_{\sigma(t)}(t)}y^*(t+1). \tag{24}$$

We observe here that the controller (21)–(24) fits into the paradigm of Section II: we set

$$\Omega=\mathcal{S}_1\times\mathcal{S}_2\times\{1,2\},$$
$$z(t)=\varnothing,$$
$$\hat{\theta}(t)=\begin{bmatrix}\hat{\theta}_1(t)\\ \hat{\theta}_2(t)\\ \sigma(t)\end{bmatrix},$$
$$r(t)=y^*(t+1).$$

In [20] it is proven that (21)–(24) provides exponential stability and a convolution bound for (20); by Theorems 1 and 2 we immediately see that the same is true in the presence of time-variation and/or unmodelled dynamics.

B. Pole-Placement Adaptive Control

In Section 8 of the pole-placement approach of [12], we proceed in a manner very similar to that of the previous sub-section, except now we are carrying out pole-placement adaptive control of a high order plant, and the algorithm which switches between the two estimators is reset every $N \geq 2n$ steps (n being the plant order). We choose $\hat{\theta}(t)$ as in the previous sub-section; however, now there is an obvious need for t and t_0 in the update law for $\sigma(t)$: we have $\sigma(t)$ constant on $[t_0, t_0 + N)$, $[t_0 + N, t_0 + 2N)$ and so on, which means that the update for that variable is a function of both t and t_0. The details are not provided due to space limitations.

Remark 6. *The robustness result proven here can also be applied to our result on high-order one-step-ahead adaptive control using a single estimator [13], [14], and to our result on step tracking in the adaptive pole-placement multi-estimator setting [21]. Space limitation prevents us from providing further details.*

Remark 7. *A reader may wonder why we have not applied Theorem 1 and 2 to other adaptive controllers in the literature. The simple answer is that, as far as the authors are aware, there are none which provides exponential stability and convolution bounds.*

VI. Summary and Conclusion

In this paper we have shown that for a class of nonlinear plant and controller combinations, if exponential stability and a convolution bound on the closed-loop behavior can be proven, then tolerance to small time-variations in the plant parameters and a small amount of unmodelled dynamics follows immediately. We applied the result to prove robustness of our recently designed multi-estimator switching adaptive controllers presented in [12] and [20]. We expect this to be applicable to other adaptive control paradigms, such as the adaptive control of nonlinear plants; this will allow one to focus on the ideal plant in the analysis, knowing that robustness will come for free. This result also has the potential to be applied in a more general nonlinear context.

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