Adaptive Set-Point Regulation using Multiple Estimators

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***Abstract*— In this paper, we consider the problem of step-tracking for an n^{th}-order discrete-time plant with unknown plant parameters belonging to a closed and bounded uncertainty set; we naturally assume that the plant does not have a zero at $z = 1$. We carry out parameter estimation for a slightly modified plant; indeed, we cover the set of admissible parameters by a finite set of compact and convex sets, and use an original-projection-algorithm based estimator for each. At each point in time, a switching algorithm is used to determine which estimates are used in the pole-placement-based controller; our approach does not assume that the switching stops at any point in time. We prove that this adaptive controller guarantees desirable linear-like closed-loop behavior (exponential stability and a bounded noise gain), as well as asymptotic tracking when the noise is constant.**

I. Introduction

Adaptive control is an approach used to deal with systems with uncertain or time-varying parameters. The first general results of adaptive control came about around 1980, e.g. [3], [4], [12], [16] and [17]. However, these controllers typically do not tolerate unmodeled dynamics, time-variations, and/or noise/disturbances very well, see e.g. [20]; furthermore, they put stringent assumptions on *a priori* information about the plant. Over the following two decades, there was a great deal of effort to address these shortcomings. A common approach was to make small controller design changes, such as σ-modifications, signal normalization, deadzones, and projection onto a convex set of admissible parameters; this provides a degree of tolerance to noise, unmodelled dynamics and/or slow time-variations, e.g. see [7], [8], [9], [15], and [23]. In general these controllers provide only asymptotic stability (and not exponential stability) with no bounded gain on the noise. Clearly, it is desirable that the closed-loop system exhibits linear-like properties, such as a bounded gain and exponential stability.

There are various non-classical approaches to adaptive control that provide linear-like system behavior. Supervisory Control [13], [14], [2] and [6] shows an efficient way to switch between candidate controllers; in certain circumstances a bounded gain on the noise is proven; here the complexity of the approach grows with the size of plant uncertainty. A falsification-based approach is discussed in [24] proving exponential stability and some form of tolerance to noise; however, some knowledge about the noise is assumed. Approaches in [18], [19] and [1] utilize multiple parameter estimators, but exponential stability is not explicitly proven.

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In [10] and [11], an approach is provided which guarantees a linear-like convolution bound on the closed-loop behavior; this yields exponential stability as well as a bounded gain on the noise. The approach employs an estimator based on the original ideal projection algorithm together with projecting parameter estimates onto a given compact and convex set. In [11], the convexity assumption is weakened (without completely removing it) and stability is proven. In [22], the same linear-like result is proven without any convexity assumption but only for the case of 1^{st}-order one-step-ahead adaptive control.

In this paper, we extend the above approach to the problem of step tracking with unknown plant parameters belonging to a closed and bounded uncertainty set; **no assumption on convexity is imposed**. We use the compactness of the set of admissible parameters to prove that it is contained in a finite union of convex sets; we run a parameter estimator for each of these sets, based on the original projection algorithm. A switching algorithm is used to determine which estimates are used in the controller; our approach does not assume that the switching stops at any point in time, but we are still able to prove linear-like behavior and asymptotic step tracking.

We use standard notation throughout the paper. We denote $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{N}$ as the sets of integers, non-negative integers and natural numbers, respectively. We will denote the Euclidean-norm of a vector and the induced norm of a matrix by the default notation $\|\cdot\|$. Also, $\boldsymbol{\ell}_\infty$ denotes the set of real-valued bounded sequences. If $\Omega \subset \mathbb{R}^p$ is a convex and compact (closed and bounded) set, we define $\|\Omega\| := \max_{x\in\Omega} \|x\|$.

II. The Setup

A. The Plant

We consider the n^{th}-order linear time-invariant discrete-time plant of the form

$$\begin{aligned} y(t+1) &= a_1 y(t) + a_2 y(t-1) + \cdots + a_n y(t-n+1) + \\ &\quad b_1 u(t) + b_2 u(t-1) + \cdots + b_n u(t-n+1) + w(t) \\ &= \phi(t)^\top \theta_{ab}^* + w(t), \quad t \in \mathbb{Z} \qquad (1) \end{aligned}$$

with $\phi(t) = \begin{bmatrix} y(t) & \cdots & y(t-n+1) & u(t) & \cdots & u(t-n+1) \end{bmatrix}^\top$ and plant parameters

$$\theta_{ab}^* = \begin{bmatrix} a_1 & a_2 & \cdots & a_n & b_1 & b_2 & \cdots & b_n \end{bmatrix}^\top \in \mathcal{S} \subset \mathbb{R}^{2n};$$

$y(t) \in \mathbb{R}$ is the measured output, $u(t) \in \mathbb{R}$ is the control input, and $w(t) \in \mathbb{R}$ is the disturbance/noise input. We assume that θ_{ab}^* is unknown but the set $\mathcal{S} \subset \mathbb{R}^{2n}$ is known. Associated with this plant model are the polynomials

$$A(z^{-1}) = 1 - a_1 z^{-1} - a_2 z^{-2} \cdots - a_n z^{-n}, \qquad \text{and}$$

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$$B(z^{-1}) = b_1 z^{-1} + b_2 z^{-2} \cdots + b_n z^{-n};$$

so the plant in (1) can be expressed in the (two-sided) z-transform form as

$$A(z^{-1})Y(z) = B(z^{-1})U(z) + z^{-1}W(z). \qquad (2)$$

We impose two assumptions on the set of admissible plant parameters.

Assumption 1. *$\mathcal{S}$ is compact, and for every $\theta \in \mathcal{S}$, the corresponding polynomials $A(z^{-1})$ & $B(z^{-1})$ are coprime.*

Here we impose no convexity assumption. The boundedness part of the assumption is reasonable in practical situations; it is used here to ensure that we can prove uniform bounds on the closed-loop behavior.

Assumption 2. *For every $\theta \in \mathcal{S}$, the corresponding polynomial $B(z^{-1})$ is such that $B(1) \neq 0$.*

The objective here is to prove an exponential form of stability, asymptotic tracking of a desired set-point $y^* \in \mathbb{R}$ and a bounded gain on the noise; the second goal clearly requires Assumption 2. Observe here that the plant may be non-minimum phase.

B. The Auxiliary Plant

If the set of admissible parameters is convex, the **classical** approach is to carry out system identification of the plant in the usual way, and do the pole placement in such a way as to force an integrator into the controller; this has been quite effective in classical results which prove asymptotic stability, e.g. see [5], as well as in our recent work [11] where we prove exponential stability. If, however, the set of admissible parameters is not convex, as is the case here, the standard trick is to replace it with its closed convex hull. Unfortunately, often that set will contain models that violate coprimeness, so another approach is sought. The compactness of the set of admissible parameters can be utilized to easily prove that it is contained in a finite union of convex sets with desirable properties; we can then use an estimator for each convex set and from time to time switch between estimates for use in the control law. In most of the results on this approach in the literature, which typically considers the disturbance-free case, to prove asymptotic tracking they generally rely on the fact that the switching mechanism stops switching at some point, e.g. [9], [6] and [1]. With unknown noise entering the system, as it is our case, it is no longer possible to conclude that switching eventually stops.

We have attempted to extend our latest work on step tracking for the convex set case given in [11, Sec. 7] to the general compact set through the use of multi-estimators; while we are able to prove an exponential type of stability, we have been unable to prove tracking[1]. In this paper we will deal with this difficulty by doing system identification on a related auxiliary model rather than the original plant model.

To proceed, let us define the tracking error by

$$\bar{y}(t) := y(t) - y^*, \qquad (3)$$

and an auxiliary control input

$$\bar{u}(t) := u(t) - u(t-1) \qquad (4)$$

as well as an adjusted disturbance signal

$$\bar{w}(t) := w(t) - w(t-1). \qquad (5)$$

If we multiply both sides of the plant model (2) by $(1-z^{-1})$ and then use the z-transformed counterpart of (4) and (5), then we end up with

$$\underbrace{(1-z^{-1})A(z^{-1})}_{=:\bar{A}(z^{-1})} Y(z) = B(z^{-1}) \underbrace{(1-z^{-1})U(z)}_{\bar{U}(z)} + z^{-1} \underbrace{(1-z^{-1})W(z)}_{\bar{W}(z)}; \qquad (6)$$

if we use the fact that $(1-z^{-1})Y^*(z) = 0$ and subtract that from the above equation, then we obtain the auxiliary plant model

$$\bar{A}(z^{-1})\bar{Y}(z) = B(z^{-1})\bar{U}(z) + z^{-1}\bar{W}(z). \qquad (7)$$

Now the polynomial $\bar{A}(z^{-1})$ has the form

$$\begin{aligned} \bar{A}(z^{-1}) &= (1-z^{-1})A(z^{-1}) \\ &= 1 - \underbrace{(1+a_1)}_{=:\bar{a}_1} z^{-1} - \underbrace{(a_2 - a_1)}_{=:\bar{a}_2} z^{-2} - \cdots \\ &\quad \cdots - \underbrace{(a_n - a_{n-1})}_{=:\bar{a}_n} z^{-n} - \underbrace{(-a_n)}_{=:\bar{a}_{n+1}} z^{-(n+1)}. \end{aligned}$$

So we see that its parameters are determined in a simple way from those of $A(z^{-1})$. Indeed, it is easy to construct an invertible matrix $\mathcal{V} \in \mathbb{R}^{(2n+1)\times(2n+1)}$ so that

$$\underbrace{\begin{bmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_{n+1} \\ b_1 \\ \vdots \\ b_n \end{bmatrix}}_{=:\theta^*} = \mathcal{V} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{bmatrix} = \mathcal{V} \begin{bmatrix} 1 \\ \theta^*_{ab} \end{bmatrix};$$

so the set of admissible parameters of (7) is given by

$$\bar{\mathcal{S}} := \left\{ \mathcal{V} \begin{bmatrix} 1 \\ \theta^*_{ab} \end{bmatrix} : \theta^*_{ab} \in \mathcal{S} \right\}. \qquad (8)$$

Using this notation, in regressor form (7) becomes

$$\bar{y}(t+1) = \bar{\phi}(t)^\top \theta^* + \bar{w}(t), \qquad (9)$$

with

$$\bar{\phi}(t) := \begin{bmatrix} \bar{y}(t) & \cdots & \bar{y}(t-n) & \bar{u}(t) & \cdots & \bar{u}(t-n+1) \end{bmatrix}^\top.$$

[1]It is interesting to observe that Morse also found it especially challenging to prove tracking in his dwell-time switching approach [14], which also occasionally experienced ongoing switching.

Remark 1. The new plant is clearly overmodelled by one variable, which is a small price to pay to achieve our objective.

Since $\mathcal{S}$ is compact, it follows that $\bar{\mathcal{S}}$ is so as well. Also, because of Assumption 1 and 2 we see that for every $\bar{\theta} \in \bar{\mathcal{S}}$, the corresponding polynomials $\bar{A}(z^{-1})$ and $B(z^{-1})$ are coprime and $B(1) \neq 0$. Of course, if we were to replace $\bar{\mathcal{S}}$ by its convex hull, then those two properties may fail to hold. This brings us to the following result, which can be used to prove that $\bar{\mathcal{S}}$ can be approximated by a finite set of convex sets which enjoy desired properties.

Proposition 1. *For every $\mu > 0$ there exist a finite number of convex, compact sets $\mathcal{S}_i \subset \mathbb{R}^{2n+1}$ that satisfy*
(i) $\bar{\mathcal{S}} \subset \bigcup_{i=1}^m \mathcal{S}_i$,
(ii) for every $\theta \in \bigcup_{i=1}^m \mathcal{S}_i$ there exists a $\bar{\theta} \in \bar{\mathcal{S}}$ that satisfy $\|\bar{\theta} - \theta\| \leq \mu$.
Furthermore, if $\mu > 0$ is sufficiently small, then we can choose the $\mathcal{S}_i$'s to have additional property as well:
(iii) for every $\theta \in \bigcup_{i=1}^m \mathcal{S}_i$, the corresponding pair of polynomials $\bar{A}(z^{-1})$ and $B(z^{-1})$ are coprime.

Proof. See the Appendix. ■

In general, finding a set of m $\mathcal{S}_i$'s which satisfy the desired properties of Proposition 1 for which m is small and $\mathcal{S}_i$ has "nice[2] structure" is not easy. However, this is not the focus of the paper. So at this point we assume that this has been done; we will show how to do this in the Example section.

To this end, the idea is to use an estimator for each $\mathcal{S}_i$, and at each point in time we choose which one to use in constructing the control law. Now define an index set $\mathcal{I}^* := \{1, 2, \ldots, m\}$. For each $\theta^* \in \bigcup_{i=1}^m \mathcal{S}_i$, we define

$$i^*(\theta^*) = \min \{i \in \mathcal{I}^* : \theta^* \in \mathcal{S}_i\};$$

when there is no ambiguity, we will drop the argument and simply write i^*. Before proceeding, define $\bar{\mathbf{s}} := \max_i \|\mathcal{S}_i\|$.

III. The Adaptive Controller

A. Parameter Estimation

Given an estimate $\hat{\theta}_i(t)$ of θ^* at time t, we define the prediction error by

$$e_i(t+1) := \bar{y}(t+1) - \bar{\phi}(t)^\top \hat{\theta}_i(t). \tag{10}$$

The common way to obtain a new parameter estimate is from solving the optimization problem

$$\operatorname*{argmin}_{\theta} \left\{ \|\theta - \hat{\theta}_i(t)\| : \bar{y}(t+1) = \bar{\phi}(t)^\top \theta \right\},$$

yielding the original projection algorithm

$$\hat{\theta}_i(t+1) = \begin{cases} \hat{\theta}_i(t) & \text{if } \bar{\phi}(t) = 0 \\ \hat{\theta}_i(t) + \frac{\bar{\phi}(t)}{\|\bar{\phi}(t)\|^2} e_i(t+1) & \text{otherwise.} \end{cases} \tag{11}$$

Of course, if $\|\bar{\phi}(t)\|$ is close to zero, numerical problem can occur, so it is the norm in the literature (e.g. see [5] and [4]) to add a constant to the denominator; however as pointed out in [10] and [11], this can lead to losing exponential stability and a bounded gain on the noise. To address this issue, as proposed in [11], with $\delta \in (0, \infty]$ we turn off the estimator if the update is larger than $2\bar{\mathbf{s}} + \delta$; first define

$$\rho_i(t) := \begin{cases} 1 & \text{if } |e_i(t+1)| < (2\bar{\mathbf{s}} + \delta)\|\bar{\phi}(t)\| \\ 0 & \text{otherwise.} \end{cases}$$

Then, with the function $\operatorname*{Proj}_{\mathcal{S}_i}\{\cdot\} : \mathbb{R}^{2n+1} \to \mathcal{S}_i$ denoting the projection onto the set $\mathcal{S}_i$, estimator updates are calculated as follows[3]:

$$\check{\theta}_i(t+1) = \hat{\theta}_i(t) + \rho_i(t) \times \frac{\bar{\phi}(t)}{\|\bar{\phi}(t)\|^2} e_i(t+1) \tag{12a}$$

$$\hat{\theta}_i(t+1) = \operatorname*{Proj}_{\mathcal{S}_i} \left\{ \check{\theta}_i(t+1) \right\}. \tag{12b}$$

Because the set $\mathcal{S}_i$ is closed and convex, the projection function is well-defined.

Define the parameter estimation error $\tilde{\theta}_i(t) := \hat{\theta}_i(t) - \theta^*$. The following lists properties of the estimation algorithm (12); these properties are the combined version of Propositions 1 and 3 of [11].

Proposition 2. *For every $t_0 \in \mathbb{Z}$, $t_2 > t_1 \geq t_0$, $\bar{\phi}(t_0) \in \mathbb{R}^{2n+1}$, $y^* \in \mathbb{R}$, $\hat{\theta}_i(t_0) \in \mathcal{S}_i$ $(i = 1, 2, \ldots, m)$, $\theta^* \in \bar{\mathcal{S}}$, and $w \in \boldsymbol{\ell}_\infty$, when the estimation algorithm in* (12) *is applied to* (9)*, the following holds:*
(i) For **every estimator***, we have*

$$\|\hat{\theta}_i(t_2) - \hat{\theta}_i(t_1)\| \leq \sum_{j=t_1}^{t_2-1} \rho_i(j) \times \frac{|e_i(j+1)|}{\|\bar{\phi}(j)\|}.$$

(ii) For the **correct estimator** *we have*

$$\|\tilde{\theta}_{i^*}(t_2)\|^2 \leq \|\tilde{\theta}_{i^*}(t_1)\|^2 + \sum_{j=t_1}^{t_2-1} \rho_{i^*}(j) \times \left[-\frac{1}{2} \frac{e_{i^*}(j+1)^2}{\|\bar{\phi}(j)\|^2} + 2 \frac{\bar{w}(j)^2}{\|\bar{\phi}(j)\|^2} \right].$$

B. Switching Control Law

The elements of $\hat{\theta}_i(t)$ can be partitioned naturally as

$$\hat{\theta}_i(t) =: [\hat{\bar{a}}_{i,1}(t) \ \cdots \ \hat{\bar{a}}_{i,n+1}(t) \ \hat{b}_{i,1}(t) \ \hat{b}_{i,2}(t) \cdots \ \hat{b}_{i,n}(t)]^\top;$$

associated with these estimates are the polynomials

$$\hat{\bar{A}}_i(t, z^{-1}) := 1 - \hat{\bar{a}}_{i,1}(t) z^{-1} - \hat{\bar{a}}_{i,2}(t) z^{-2} \cdots - \hat{\bar{a}}_{i,n+1}(t) z^{-(n+1)},$$
$$\hat{B}_i(t, z^{-1}) := \hat{b}_{i,1}(t) z^{-1} + \hat{b}_{i,2}(t) z^{-2} \cdots + \hat{b}_{i,n}(t) z^{-n}.$$

Next we design a $(n+1)^{\text{th}}$-order strictly proper controller; we choose the following polynomials

$$\hat{L}_i(t, z^{-1}) = 1 + \hat{l}_{i,1}(t) z^{-1} + \hat{l}_{i,2}(t) z^{-2} + \cdots + \hat{l}_{i,n}(t) z^{-n},$$
$$\hat{P}_i(t, z^{-1}) = \hat{p}_{i,1}(t) z^{-1} + \hat{p}_{i,2}(t) z^{-2} + \cdots + \hat{p}_{i,n+1}(t) z^{-(n+1)}$$

to place all closed-loop poles at $z = 0$:

$$\hat{\bar{A}}_i(t, z^{-1}) \hat{L}_i(t, z^{-1}) + \hat{B}_i(t, z^{-1}) \hat{P}_i(t, z^{-1}) = 1. \tag{13}$$

[2]Nice in the sense that it is computationally easy to project onto it.

[3]In case of $\delta = \infty$, we will adopt the understanding that $\infty \times 0 = 0$, in which case this formula collapses to the original projection algorithm (11).

Given the assumption that the $\hat{\hat{A}}_i(t,z^{-1})$ and $\hat{B}_i(t,z^{-1})$ are coprime, we know that there exist unique $\hat{L}_i(t,z^{-1})$ and $\hat{P}_i(t,z^{-1})$ which satisfy this equation; it is also easy to prove that the coefficients of $\hat{L}_i(t,z^{-1})$ and $\hat{P}_i(t,z^{-1})$ are analytic functions of $\hat{\theta}_i(t) \in \mathcal{S}_i$. For a **suitable choice of** $i \in \mathcal{I}^*$ **at time** t, we define the control input by

$$\hat{L}_i(t-1,z^{-1})\bar{U}(z) = -\hat{P}_i(t-1,z^{-1})\bar{Y}(z). \qquad (14)$$

This can be written in terms of the state vector: to proceed we define the control gains $\hat{K}_i(t) \in \mathbb{R}^{2n+1}$:

$$\hat{K}_i(t) := \begin{bmatrix} -\hat{p}_{i,1}(t) & \cdots & -\hat{p}_{i,n+1}(t) & -\hat{l}_{i,1}(t) & \cdots & -\hat{l}_{i,n}(t) \end{bmatrix} \qquad (15)$$

so that (14) becomes

$$\bar{u}(t) = \hat{K}_i(t-1)\bar{\phi}(t-1).$$

We will use a switching signal $\sigma : \mathbb{Z} \to \mathcal{I}^*$ which determines the index i at any given point in time.

In our earlier work [11], we considered the problem of closed-loop stability (but not tracking) in the case of switching between 2 estimators. Unfortunately, the approach does not extend in a simple way to the case of $m > 2$ estimators, so we will need a new algorithm. Our closed-loop system behavior will in large part be determined by a time-varying matrix $\mathcal{A}_{\sigma(t)}(t) \in \mathbb{R}^{(2n+1)\times(2n+1)}$; at all times this matrix will be deadbeat, i.e. all of its eigenvalues will be at zero. However, its product

$$\mathcal{A}_{\sigma(t)}(t) \times \mathcal{A}_{\sigma(t-1)}(t-1) \times \cdots \times \mathcal{A}_{\sigma(t_0)}(t_0), \qquad t \geq t_0$$

will not usually be deadbeat. A natural solution to this problem is to update the estimators every $2n+1$ steps; the problem with this idea is that we end up with no information about $e_i(t+1)$ between the updates, so the closed-loop system is not amenable to analysis. So our solution procedure will need to be different: we update $\sigma(t)$ only every $N \geq 2n+1$ steps; however, we keep the estimators running and the control gains updating. To this end, we define a sequence of switching times as follows: we initialize $\hat{t}_0 := t_0$ and then define

$$\hat{t}_\ell := t_0 + \ell N, \ \ell \in \mathbb{N}.$$

So the switching signal is set as

$$\sigma(t) = \sigma(\hat{t}_\ell), \qquad t \in [\hat{t}_\ell, \hat{t}_{\ell+1}), \ \ell \in \mathbb{Z}^+. \qquad (16)$$

We now define the control law as

$$\bar{u}(t) = \hat{K}_{\sigma(t-1)}(t-1)\bar{\phi}(t-1), \qquad t > t_0; \qquad (17)$$

and from (4), the plant control input is

$$u(t) = \bar{u}(t) + u(t-1), \qquad t > t_0. \qquad (18)$$

What remains to be defined is the choice of the switching signal $\sigma(\hat{t}_\ell)$, which we will do in the next subsection.

C. Switching Algorithm

Define the set of switching times as $\mathcal{T}_N := \left\{\hat{t}_\ell \geq t_0 : \hat{t}_\ell = t_0 + \ell N, \ \ell \in \mathbb{Z}^+\right\}$. To proceed , we define a performance signal $J_i : \mathcal{T}_N \to \mathbb{R}^+$ for each estimator $i \in \mathcal{I}^*$; for $\ell \in \mathbb{Z}^+$, we define

$$J_i(\hat{t}_\ell) := \sum_{j=\hat{t}_\ell}^{\hat{t}_{\ell+1}-1} \rho_i(j) \times \frac{|e_i(j+1)|}{\|\bar{\phi}(j)\|}; \qquad (19)$$

this quantity is an upper bound on the amount of change in $\hat{\theta}_i(t)$ on the interval $[\hat{t}_\ell, \hat{t}_{\ell+1})$. Clearly, the estimator with the least amount of update should be the best one, which will lead to a switching signal of the form

$$\sigma(\hat{t}_{\ell+1}) = \underset{i \in \mathcal{I}^*}{\text{argmin}} \ J_i(\hat{t}_\ell).$$

Although this rule works in every simulation that we try, the proof remains elusive; a potential problem is that the switching signal could oscillate between two bad choices, and never (or rarely) choose a "correct" one. Instead, we propose a different approach. At each switching time $\hat{t}_\ell$ we have an admissible set $\mathcal{I}(\hat{t}_\ell)$: we initialize $\mathcal{I}(\hat{t}_0) = \mathcal{I}^*$, and we obtain $\mathcal{I}(\hat{t}_{\ell+1})$ from $\mathcal{I}(\hat{t}_\ell)$ by removing all $j \in \mathcal{I}(\hat{t}_\ell)$ satisfying

$$J_{\sigma(\hat{t}_\ell)}(\hat{t}_\ell) \leq J_j(\hat{t}_\ell);$$

clearly $j = \sigma(\hat{t}_\ell)$ satisfies this bound, but more j's may as well; if this results in $\mathcal{I}(\hat{t}_{\ell+1})$ being empty, then we **reset** $\mathcal{I}(\hat{t}_{\ell+1})$ to be $\mathcal{I}^*$. This **Switching Algorithm** is summarized in the following: with $\sigma(\hat{t}_0) = \sigma_0$ and $\mathcal{I}(\hat{t}_0) = \mathcal{I}^*$, for $\ell \in \mathbb{Z}^+$:

$$\hat{\mathcal{I}}(\hat{t}_\ell) = \left\{ i \in \mathcal{I}^* \ : \ J_i(\hat{t}_\ell) < J_{\sigma(\hat{t}_\ell)}(\hat{t}_\ell) \right\}, \qquad (20a)$$

$$\mathcal{I}(\hat{t}_{\ell+1}) = \begin{cases} \mathcal{I}^* & \text{if } \mathcal{I}(\hat{t}_\ell) \cap \hat{\mathcal{I}}(\hat{t}_\ell) = \varnothing \\ \mathcal{I}(\hat{t}_\ell) \cap \hat{\mathcal{I}}(\hat{t}_\ell) & \text{otherwise,} \end{cases} \qquad (20b)$$

$$\sigma(\hat{t}_{\ell+1}) = \underset{i \in \mathcal{I}(\hat{t}_{\ell+1})}{\text{argmin}} \ J_i(\hat{t}_\ell), \qquad (20c)$$

i.e. we keep all models in the admissible index set for which the performance signal is better (i.e. smaller) than the one we are currently using.

Lemma 1. *Consider the plant* (9)*, and suppose that the controller* (12) *and* (15)–(20) *is applied. Then for every* $t_0 \in \mathbb{Z}$*,* $y^* \in \mathbb{R}$*,* $\sigma_0 \in \mathcal{I}^*$*,* $\bar{\phi}(t_0) \in \mathbb{R}^{2n+1}$*,* $\theta^* \in \bar{\mathcal{S}}$*,* $N \geq 1$*,* $\hat{\theta}_i(t_0) \in \mathcal{S}_i$ *(*$i \in \mathcal{I}^*$*), and* $w \in \ell_\infty$*, if* $\hat{t}_{\underline{\ell}}$ *and* $\hat{t}_{\overline{\ell}}$ *are two consecutive reset times of the index set, then there exists a* $\ell^* \in [\underline{\ell}, \overline{\ell})$ *such that:*

$$J_{\sigma(\hat{t}_{\ell^*})}(\hat{t}_{\ell^*}) \leq J_{i^*}(\hat{t}_{\ell^*}).$$

In the above we do not make any claim that $\theta^* \in \mathcal{S}_{\sigma(t)}$ at any time; it only makes an indirect statement about the size of the prediction error. It turns out that this is enough to ensure that desired closed-loop behavior is attained.

IV. THE MAIN RESULT

In most adaptive controllers the goal is to prove asymptotic results, so details of initial conditions is not important. However, we want to get a bound on the transient behavior. With a starting time of t_0, define the initial condition as

$$\boldsymbol{\phi}_0 := \begin{bmatrix} y(t_0) & \cdots & y(t_0-n) & u(t_0) & \cdots & u(t_0-n) \end{bmatrix}^\top.$$

Theorem 1. *Consider the n^{th}-order plant* (1) *satisfying Assumptions 1 and 2 with the corresponding $\{\mathcal{S}_i\}_{i=1}^m$ satisfying the conditions of Proposition 1. Suppose that the controller* (12), (15)–(20) *is applied. For every $\lambda \in (0,1)$, $\delta \in (0,\infty]$ and $N \geq 2n+1$, there exists a constant $\gamma > 0$ such that for every $t_0 \in \mathbb{Z}$, $\phi_0 \in \mathbb{R}^{2n+2}$, $\sigma_0 \in \mathcal{I}^*$, $\theta^*_{ab} \in \mathcal{S}$, $\hat{\theta}_i(t_0) \in \mathcal{S}_i$ ($i \in \mathcal{I}^*$), $y^* \in \mathbb{R}$ and $w \in \boldsymbol{\ell}_\infty$, the following holds*

$$\|\phi(t)\| \leq \gamma\lambda^{t-t_0}\|\phi_0\| + \gamma|y^*| + \gamma\sum_{j=t_0}^{t-1}\lambda^{t-1-j}|w(j)|,\ t \geq t_0;$$

furthermore, if $w(\cdot)$ is constant, then

$$\lim_{t\to\infty} y(t) = y^*.$$

The above result shows that the closed-loop system experiences linear-like behavior. There is a uniform exponential decay bound on the effect of the initial condition, and a convolution sum bound on the effect of the noise. This implies that the system has a bounded gain (from w and y^* to y) in every p-norm: for $p = \infty$, we can see from the above bound that

$$\|\phi(t)\| \leq \gamma(\|\phi_0\| + |y^*|) + \frac{\gamma}{1-\lambda}\sup_{j\in[t_0,t)}|w(j)|.$$

We emphasize here that we are able to show the result using a switching control law without assuming that the switching stops. As far as the authors know, only a few similar results are found in the literature, e.g. Morse [14] and [2], although convolution bounds are not proven.

Proof of Theorem 1. Due to space limitation, only a brief sketch of the proof is outlined here. The proof uses similar, but not identical, analysis to that used in [22] and [11, Sec. 8], which, in turn, is based on the analysis in [10]. Recalling the compactness of the $\mathcal{S}_i$'s and the definition of switching times $\hat{t}_\ell$, the steps of the proof go as follows:

1) first, from (10) and (17) we obtain a state-space equation describing $\bar{\phi}(t)$ which holds on intervals of the form $[\hat{t}_\ell, \hat{t}_{\ell+1})$;
2) second, utilizing the deadbeat nature of the equation, the nature of the estimation process and Proposition 2(i), we analyze this equation to get a bound on $\|\bar{\phi}(\hat{t}_{\ell+1})\|$ in terms of $\|\bar{\phi}(\hat{t}_\ell)\|$;
3) applying Lemma 1, we obtain a bound on $\bar{\phi}$ between index set resets of the Switching Algorithm (20);
4) next, by Proposition 2(ii) and Lemma 2 of [10], we analyze the associated difference inequality, relating the behavior between index set resets, to obtain a general bound on $\bar{\phi}(t)$;
5) from the previous step, we directly get a bound on y and prove asymptotic tracking. We then obtain a bound on u based on the observability of all admissible plants (Assumption 1) to then finally get the desired bound on $\phi(t)$. ∎

Remark 2. Utilizing the linear-like convolution bounds found in Theorem 1, the result can be extended to prove tolerance to parameter time-variation & unmodelled dynamics; the proof uses the same approach as the proofs of Theorem 2 and 3 in [11] for the case of a convex uncertainty set.

V. A Simulation Example

In this section, a simulation example is provided to illustrate the results of this paper. Consider the 2nd-order plant:

$$y(t+1) = a_1(t)y(t) + a_2(t)y(t-1) + b_1(t)u(t) + b_2(t)u(t-1) + w(t)$$

with parameters belonging to the uncertainty set $\mathcal{S}$:

$$\mathcal{S} := \Big\{ \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \end{bmatrix}^\top \in \mathbb{R}^4 : a_1 \in [-2,0], a_2 \in [-3,-1], b_1 \in [-1,0], b_2 \in [-5,-3] \cup [3,5] \Big\}.$$

Hence, every admissible model is unstable and non-minimum phase, which makes this plant challenging to control; it has two complex unstable poles together with a zero that can lie in $[3,\infty)$. It is also obvious to see that $\mathcal{S}$ is not a convex set; notice that the convex hull of it includes the case of having $b_1 = b_2 = 0$, which corresponds to a non-stabilizable system, violating the coprimeness assumption. So, we apply the proposed approach in this paper.

We define the set $\bar{\mathcal{S}}$ by (8); so we will be estimating the parameters of the auxiliary plant: $\theta^*(t) = \begin{bmatrix} \bar{a}_1(t) & \bar{a}_2(t) & \bar{a}_3(t) & b_1(t) & b_2(t) \end{bmatrix}^\top \in \bar{\mathcal{S}}$. We know that the set $\bar{\mathcal{S}}$ is also compact and satisfies the coprimeness requirement; we will need to find a set of convex and compact sets that their union contains $\bar{\mathcal{S}}$ and that will also satisfy the coprimeness requirement. We define

$$\mathcal{S}_1 := \Big\{ \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & b_1 & b_2 \end{bmatrix}^\top \in \mathbb{R}^5 : \bar{a}_1 \in [-1,1], \bar{a}_2 \in [-3,1], \bar{a}_3 \in [1,3], b_1 \in [-1,0], b_2 \in [-5,-3] \Big\},$$

$$\mathcal{S}_2 := \Big\{ \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & b_1 & b_2 \end{bmatrix}^\top \in \mathbb{R}^5 : \bar{a}_1 \in [-1,1], \bar{a}_2 \in [-3,1], \bar{a}_3 \in [1,3], b_1 \in [-1,0], b_2 \in [3,5] \Big\}.$$

Each of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is a hyperrectangle, which is easy to project onto; we easily see that $\bar{\mathcal{S}} \subset \mathcal{S}_1 \cup \mathcal{S}_2$. We can also verify that each of $\mathcal{S}_1$ and $\mathcal{S}_2$ contain models that are coprime, as desired.

For this simulation we set $a_1 = -\frac{1}{2}$, $a_2 = -\frac{3}{2}$, $b_1 = -\frac{3}{4}$, and $b_2 = -3$. We will apply the proposed controller (12) and (15)–(20); we choose $N = 5$ and $\delta = \infty$. We set the desired set-point $y^* = 2$, and initial condition $y(0) = y(-1) = y(-2) = -1$ and $u(-1) = u(-2) = 0$; we also set $\hat{\theta}_1(0) = \begin{bmatrix} 0 & -1 & 2 & -\frac{1}{2} & -4 \end{bmatrix}^\top$, $\hat{\theta}_2(0) = \begin{bmatrix} 0 & -1 & 2 & -\frac{1}{2} & 4 \end{bmatrix}^\top$ and $\sigma_0 = 2$. We set the disturbance to be of a constant magnitude: $|w(t)| = \frac{1}{2}$, but with its sign changing every 250 steps. Figure 1 displays the results. We see that the controller does a good job of tracking; the closed-loop system experiences some transient behavior when the disturbance changes, but the tracking recovers quickly.

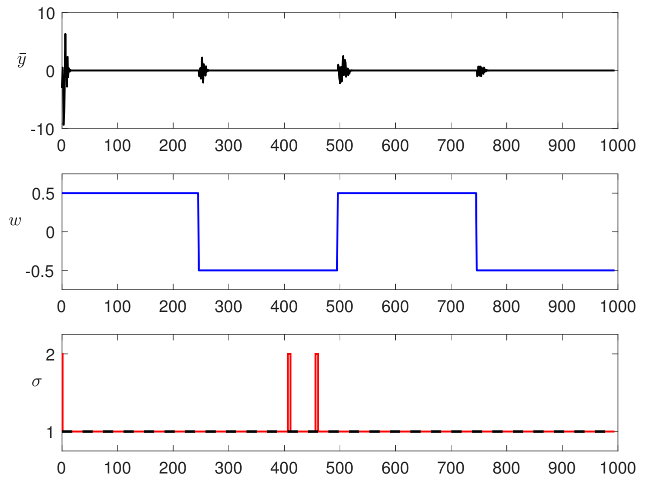

Fig. 1. The top plot shows the tracking error $\bar{y}$; the middle plot shows the disturbance w; the bottom plot shows the switching signal σ (solid) and the correct index i^* (dashed).

VI. Conclusion

In this paper, we consider the problem of step-tracking of an n^{th}-order plant with unknown plant parameters belonging to a compact uncertainty set; no assumption of convexity is imposed. We cover the set of admissible parameters by a finite set of compact and convex sets, and use a projection-algorithm-based parameter estimator for each one. A switching algorithm is used to determine which estimates are used at each point in time. We prove that this adaptive controller guarantees linear-like convolution bounds on the closed-loop behavior, which implies exponential stability and a bounded noise gain; **this is rarely found in the literature**. Furthermore, when the noise is constant, we prove that asymptotic tracking is achieved.

We would like also to extend the approach to the tracking of more general reference signals, and to include cases where the order of the plant is unknown.

Appendix

Proof of Proposition 1. Fix $\mu > 0$. For every $x \in \bar{\mathcal{S}}$, let $\mathcal{O}_x \subset \mathbb{R}^{2n+1}$ denote the open ball of radius μ centered at x. Then $\{\mathcal{O}_x : x \in \bar{\mathcal{S}}\}$ is an open cover of $\bar{\mathcal{S}}$, so by the Heine-Borel Theorem [21] there exist $x_1, x_2, \ldots, x_m$ so that $\bar{\mathcal{S}} \subset \bigcup_{i=1}^m \mathcal{O}_{x_i}$. If we set $\mathcal{S}_i :=$ closure of $\mathcal{O}_{x_i}$, then (i) and (ii) of the required properties hold.

If $\bar{A}(z^{-1})$ and $B(z^{-1})$ are the corresponding polynomials associated with $x \in \mathbb{R}^{2n+1}$, then let $\mathbb{S}(x) \in \mathbb{R}^{(2n+1)\times(2n+1)}$ denote the Sylvester Matrix associated with the pair of polynomials (see [5, p. 482]). By the coprimeness requirement, we know that $\min_{\bar{\theta}\in\bar{\mathcal{S}}} |\det \mathbb{S}(\bar{\theta})| > 0$. As $\det \mathbb{S}(x)$ is continuous in x, if a small enough $\mu > 0$ is used in the procedure (of the previous paragraph) to construct the $\mathcal{S}_i$'s, we conclude that $\min_{\theta\in\bigcup_{i=1}^m \mathcal{S}_i} |\det \mathbb{S}(\theta)| > 0$. ■

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